Supplementary Notes for ELEN 4810 Lecture 4
Sampling and "C/D Conversion"

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Disclaimer: These notes are intended to be an accessible introduction to the subject, with no pretense at completeness. In general, you can find more thorough discussions in Oppenheim's book. Please let me know if you find any typos.

Reading suggestions: Oppenheim and Schafer Sections 4.1-4.6

In this lecture, we will first complete Lecture 3 material around the properties of the Discrete-Time Fourier Transform. In these lecture notes, we will use this tool to study the relationship between a continuous-time signal and its sampled (discrete-time) variant. We will discuss a fundamental result, known as the Shannon-Nyquist Sampling Theorem, which tells us that *bandlimited* continuous-time signals can be perfectly reconstructed from their samples.

1 The sampling theorem

Consider a continuous-time signal,

$$x_c(t) \qquad t \in \mathbb{R}. \tag{1.1}$$

We obtain samples of this signal by taking its value every T seconds:

$$x[n] = x_c(nT), \qquad n \in \mathbb{Z}. \tag{1.2}$$

How do the samples $x[\cdot]$ relate to the continuous signal $x_c(\cdot)$? What information have we lost when we restrict our attention to $x[\cdot]$?

These questions are most naturally answered in the Fourier domain. Our approach in this lecture will be mathematical in nature – it will describe how, under idealized assumptions, once can recover $x_c(t)$ from $x[n]$. The sampling and reconstruction systems we describe in this lecture should be viewed as mathematical idealizations – useful for understanding the fundamental limits of sampling and reconstruction, but only loose approximations to what can actually be implemented. In subsequent lectures, we will give a small taste of the practical considerations that go into implementing real A/D converters.

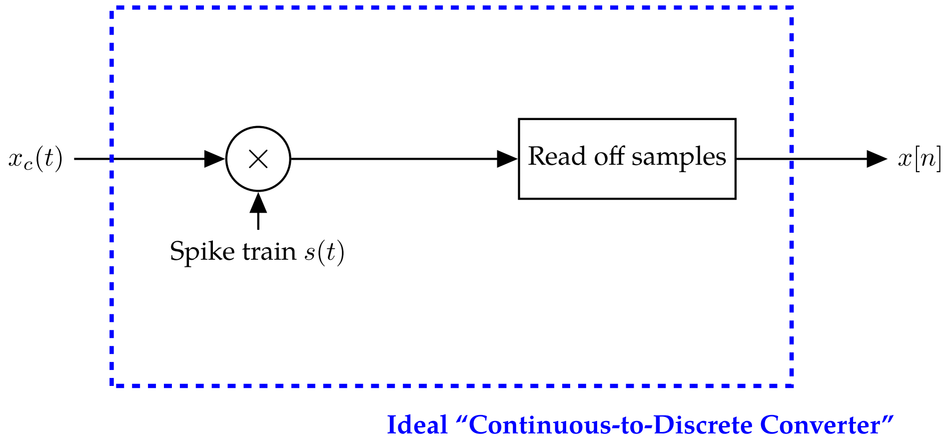

Figure 1: **Mathematical Model for Sampling.** This model aids in deriving frequency domain relationships between $x_c(t)$ and $x[n]$. In the model, $x_c(t)$ is first modulated by the spike train $s(t) = \sum_n \delta(t - nT)$, producing a spike train $x_s(t) = \sum_n x_c(nT)\delta(t - nT)$. From thjis spike train, we read off the samples $x[n] = x_c(nT)$.

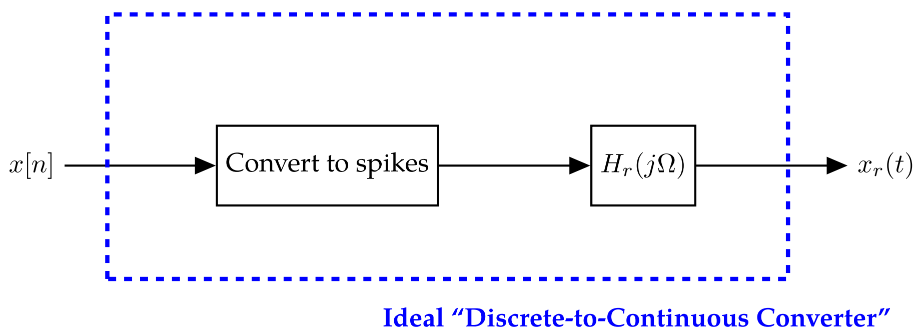

Figure 2: **Mathematical Model for Reconstruction from Samples.** Here, the discrete-time signal $x[n]$ is first converted to a continuous-time spike train $x_s(t) = \sum_n x[n]\delta(t - nT)$, which is then passed through an ideal recontruction filter with frequency response $H_r(j\Omega) = \begin{cases} T & |\Omega| \leq \frac{\pi}{T} \\ 0 & \text{else} \end{cases}$.

We will describe a mathematical representation of the sampling process, which happens in two steps. We first introduce another signal $s(t)$, which represents our sampling locations with a sequence of Dirac delta measures:

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT). \tag{1.3}$$

This signal is sometimes called a "periodic impulse train" or "Dirac comb". Its (continuous time) Fourier transform is also an impulse train:

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s), \qquad \Omega_s = \frac{2\pi}{T} \tag{1.4}$$

If we form the product $x_s(t) = s(t)x_c(t)$, then by the sifting property of the Dirac delta,

$$\begin{aligned} x_s(t) &= \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT) && (1.5)\\ &= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT) && (1.6)\\ &= \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT). && (1.7) \end{aligned}$$

So, an idealized version of the sampling process can be described as follows: we first modulate the signal by multiplying by $s(t)$. This creates an impulse train whose amplitudes are the samples $x[n]$. We then read off the amplitudes to create the discrete time signal $x[n]$.

To understand the effect of these operations, let us use the fact that for the (continuous) Fourier transform, modulation in time domain is equivalent to convolution in frequency domain to express

$$\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) && (1.8)\\ &= \frac{1}{2\pi} \int_{\Omega'=-\infty}^{\infty} X_c(j\Omega') S(j(\Omega - \Omega')) d\Omega' && (1.9)\\ &= \frac{1}{2\pi} \int_{\Omega'=-\infty}^{\infty} X_c(j\Omega') \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\Omega - \Omega' - k\Omega_s) d\Omega' && (1.10)\\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} X_c(j\Omega')\delta(\Omega - \Omega' - k\Omega_s) d\Omega' && (1.11)\\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)). && (1.12) \end{aligned}$$

That is to say, the Fourier transform of the modulated signal $X_s(j\Omega)$ is a superposition of shifted copies $X_c(j(\Omega - k\Omega_s))$ of the Fourier transform X_c of the continuous input $x_c(t)$.

It is instructive to draw pictures of this process in two cases: first, when the input is *band limited* with bandlimit Ω_N:

$$X_c(j\Omega) = 0, \qquad |\Omega| \geq \Omega_N. \tag{1.13}$$

If $\Omega_s \geq 2\Omega_N$, then the shifted copies of X_c in (1.12) do not overlap. In particular, it is possible to "pick out" the copy corresponding to $k = 0$ by forming the ideal (low-pass) reconstruction filter

$$H_r(j\Omega) = \begin{cases} T & |\Omega| \leq \Omega_s/2 \\ 0 & \text{else.} \end{cases} \tag{1.14}$$

When $\Omega_s \geq 2\Omega_N$, we have

$$H_r(j\Omega)X_s(j\Omega) = X_c(j\Omega). \tag{1.15}$$

In this situation, the following *conceptual* reconstruction procedure reproduces $x_c(t)$ exactly from the samples $x[n]$: given $x[n]$, we first form the impulse train $x_s(t)$, and then convolve it with the ideal reconstruction filter $h_r(t)$, defined via (1.14). When $\Omega_s \geq 2\Omega_N$ (and hence $H_r(j\Omega)$ correctly selects only a single copy of $X_c(j\Omega)$), this conceptual procedure reproduces $x_c(t)$ exactly. This argument, made slightly more formal, gives a result known as the *Shannon-Nyquist*[1] sampling theorem:

Theorem 1.1. *A bandlimited continuous-time signal $x_c(t)$, with bandlimit Ω_N, can be exactly reconstructed from its samples $x[n] = x_c(nT)$, provided the sampling rate $\Omega_s = 2\pi/T$, satisfies $\Omega_s \geq 2\Omega_N$.*

As a maxim: *to exactly reconstruct a bandlimited signal, sample at a rate which is greater than twice the highest frequency*. This sampling rate is called the *Nyquist rate*.

What happens when the signal is not band limited, or we sample below the Nyquist rate? Again, it is instructive to draw a picture. In this case, the shifted copies $X_c(j(\Omega - k\Omega_s))$ may overlap, and low-pass filtering with the ideal reconstruction filter H_r will not produce $X_c(j\Omega)$. For example, following the text, suppose that $x_c(t)$ is a sinusoid:

$$x_c(t) = \cos(\Omega_0 t). \tag{1.16}$$

The Fourier transform of x_c is $X_c(j\Omega) = \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0)$. Hence, exact reconstruction demands that we sample at a rate of $\Omega_s > 2\Omega_0$. Sampling at lower rates will cause the shifted spike $\delta(\Omega + \Omega_0 - \Omega_s)$ to occur at $\Omega_s - \Omega_0$, to the left of the spike $\delta(\Omega - \Omega_0)$. In the reconstruction $x_r(t) = h_r(t) * x_s(t)$, this creates an artificial component with frequency $\Omega_s - \Omega_0$ – a frequency that was not present in the original signal! This is an example of a broader phenomenon known as *aliasing*: because there are infinitely many sinusoids consistent with a given discretized sinusoidal signal, after sampling, high-frequency sinusoids can masquerade as lower-frequency sinusoids, leading to incorrect reconstructions. The Shannon-Nyquist theorem tells us that *if* the signal is band limited and we have sampled at a sufficiently high rate, we can avoid these ambiguities, simply by low-pass filtering.

This phenomenon is best understood graphically. Figures 3, 4 and 5 show three examples of what can happen. In Figure 3 we sample above the Nyqvist rate, and so the copies $X_c(\Omega - k\Omega_s)$ of the spectrum of the original signal do not overlap. Likewise, in Figure 4, we sample at the Nyqvist rate, and so the copies $X_c(\Omega - k\Omega_s)$ of the spectrum do not overlap. However, in Figure 5, we sample below the Nyqvist rate, and aliasing occurs: copies of the spectrum *do* overlap.

[1]Or Shannon-Nyquist-Kotelnikov – Kotelnikov actually had the result in 1933, a number of years before Shannon's publication.

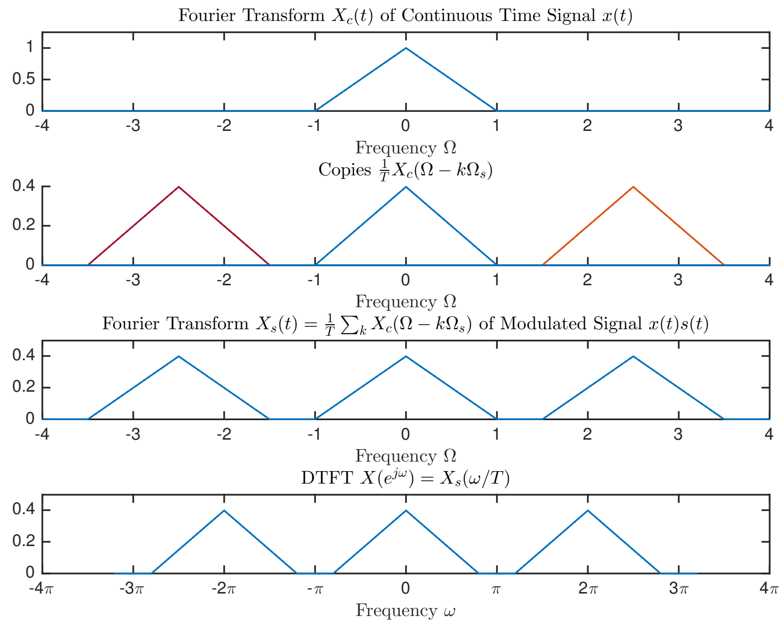

Figure 3: **Sampling above the Nyquist rate.** Top: Fourier transform of original continuous-time signal $x_c(t)$, with bandlimit $\Omega_N = 1$. Next: copies $\frac{1}{T}X_c(\Omega - k\Omega_s)$, with $\Omega_s = 1.25 \times 2\Omega_N$. The Fourier transform of the modulated signal $x_c(t)s(t)$ does not exhibit aliasing (distinct copies of X_c do not overlap. Bottom: the DTFT of the sampled signal $x[n]$ consists of 2π-periodized copies of the original spectrum.

Reconstructing $x_c(t)$ from samples. The sampling theorem tells us that if x_c is band limited and we sample at or above the Nyquist rate, we will be able to exactly reconstruct x_c from its samples. The arguments we presented above in favor of this viewpoint actually suggest the reconstruction procedure – form $x_s(t)$ and then apply the ideal reconstruction filter $h_r(t)$. In frequency domain, H_r is simply a scaled box function. Recalling (or rederiving) that the inverse Fourier transform of a box function is a sinc, we obtain the following expression for $h_r(t)$:[2]

$$h_r(t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t/T)}{\pi t/T} & \text{else} \end{cases} \tag{1.17}$$

[2]This is usually just written $h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}$ with the understanding that one should evaluate at $t = 0$ by taking the limit as t approaches zero.

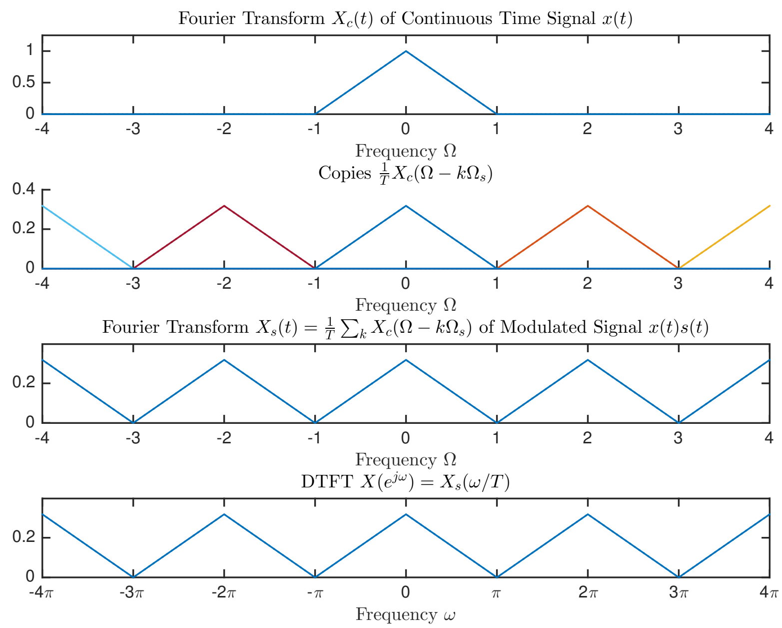

Figure 4: **Sampling at the Nyquist rate.** Top: Fourier transform of original continuous-time signal $x_c(t)$, with bandlimit $\Omega_N = 1$. Next: copies $\frac{1}{T}X_c(\Omega - k\Omega_s)$, with $\Omega_s = 2\Omega_N$. The Fourier transform of the modulated signal $x_c(t)s(t)$ does not exhibit aliasing (distinct copies of X_c do not overlap. Bottom: the DTFT of the sampled signal $x[n]$ consists of 2π-periodized copies of the original spectrum.

Hence, the reconstructed signal is

$$
\begin{aligned}
x_r(t) &= x_s(t) * h_r(t) && (1.18)\\
&= \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT) && (1.19)\\
&= \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}. && (1.20)
\end{aligned}
$$

It is worth noting that $h_r(0) = 1$, and $h_r(kT) = 0$ for any integer $k \neq 0$. Thus, for every integer n,

$$x_r(nT) = x[n]. \tag{1.21}$$

So, the ideal reconstruction reproduces each of the sample values, and *interpolates* in between the sample values using sinc functions. The (remarkable) implication of the sampling theorem is that this interpolation is exact, when the input is band limited.

Frequency content of the sampled signal. Often, we sample signals with the goal of performing some sort of processing on them in the discrete domain. For this purpose, it is useful to know how the frequency content of the sampled signal $x[n]$ relates to that of the continuous domain signal

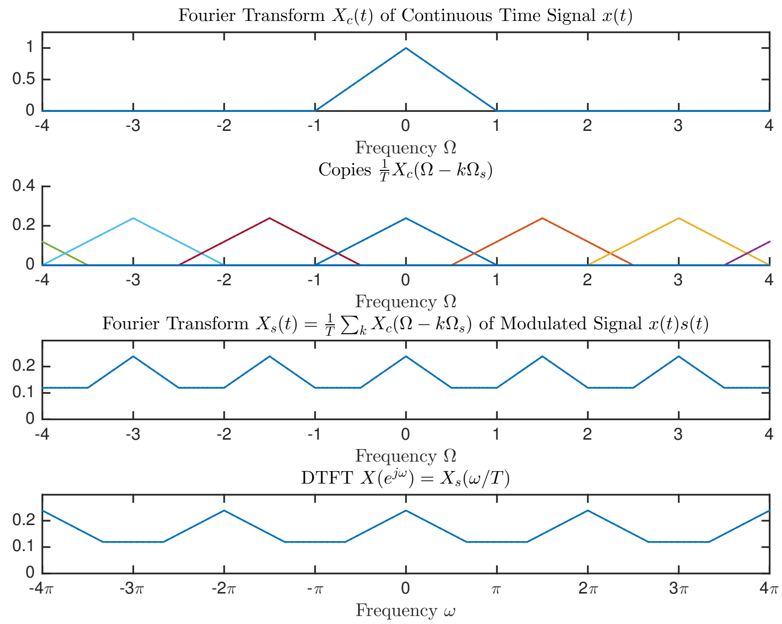

Figure 5: **Sampling below the Nyquist rate.** Top: Fourier transform of original continuous-time signal $x_c(t)$, with bandlimit $\Omega_N = 1$. Next: copies $\frac{1}{T}X_c(\Omega - k\Omega_s)$, with $\Omega_s = 2\Omega_N$. The Fourier transform of the modulated signal $x_c(t)s(t)$ does exhibit aliasing (distinct copies of X_c overlap. Bottom: the DTFT of the sampled signal $x[n]$ does not directly exhibit the spectrum of the original continuous time signal.

$x_c(t)$. To this end, note that

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)), \tag{1.22}$$

and that

$$\begin{aligned} X_s(j\Omega) &= \int_{t=-\infty}^{\infty} \sum_n x[n]\delta(t-nT)\exp(-j\Omega t)\ dt & (1.23)\\ &= \sum_{n=-\infty}^{\infty} x[n]\int_{t=-\infty}^{\infty} \delta(t-nT)\exp(-j\Omega t)\ dt & (1.24)\\ &= \sum_{n=-\infty}^{\infty} x[n]\exp(-j\Omega T n) & (1.25)\\ &= X(e^{j\omega})\big|_{\omega=\Omega T}. & (1.26)\end{aligned}$$

That is to say, the (continuous-time) Fourier transform of the impulse train x_s and the Discrete-Time Fourier Transform of the discrete-time signal $x[n]$ are simply related by a scaling of frequency:

$\omega = \Omega T$. From this and our expression for X_s, we get

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]. \tag{1.27}$$

In particular, if the sampling rate is high enough that there is no aliasing, the discrete time Fourier transform of $x[n]$ simply consists of shifted copies of the Fourier transform of x_c, subject to the rescaling $\omega = \Omega T$.

2 Processing continuous time signals with discrete-time tools

The above discussion suggests that when the input is band limited and the sampling rate is high enough (above the Nyquist rate), the frequency domain representations of $x[n]$ and $x_c(t)$ coincide, up to a scaling by $1/T$ and a stretching of the frequency axis via the equation $\omega = \Omega T$. So, it is tempting to believe that we can perform operations on the continuous time signal, by working in discrete time and then applying the ideal reconstruction filter to reproduce a continuous time output.

Indeed, let us perform the following sequence of operations:

- **Sample**: Set $x[n] = x_c(nT)$.
- **Filter in discrete time**: Set $y = x * h$, for some discrete-time system's impulse response $h[n]$.
- **Reconstruct**: Set $y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}$.

In frequency domain,

$$\begin{aligned} Y_r(j\Omega) &= H_r(j\Omega)Y(e^{j\Omega T}) && (2.1)\\ &= \begin{cases} TY(e^{j\Omega t}) & |\Omega| < \pi/T \\ 0 & \text{else.} \end{cases} && (2.2) \end{aligned}$$

Moreover, $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$. So

$$\begin{aligned} Y_r(j\Omega) &= H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T}) && (2.3)\\ &= H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\Omega - \frac{2\pi k}{T}\right)\right]. && (2.4) \end{aligned}$$

If x_c is band limited, so $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$, then

$$\begin{aligned} Y_r(j\Omega) &= \begin{cases} H(e^{j\Omega T})X_c(j\Omega) & |\Omega| \leq \pi/T \\ 0 & \text{else} \end{cases} && (2.5)\\ &= H_{\text{eff}}(j\Omega)X_c(j\Omega), && (2.6) \end{aligned}$$

with

$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}) & |\Omega| < \pi/T \\ 0 & \text{else} \end{cases} \tag{2.7}$$

So, finally, *for band limited inputs*, the composite system acts like a linear time invariant system, whose frequency response is simply $H_{\text{eff}}(j\Omega)$. You may wish to think carefully about what happens when the input is not band limited? Is the overall system still LTI?

We can use these relationships to design a discrete time system which, when applied to samples, produces a desired continuous-time effect. Indeed, suppose we want to apply a continuous time LTI system with frequency response $H_c(j\Omega)$. Inspired by our above expression for $H_{\text{eft}}(j\Omega)$, let us demand that

$$\begin{aligned} H(e^{j\omega}) &= H_c(j\omega/T) \qquad |\omega| < \pi & (2.8) \\ H_c(j\Omega) &= 0 \qquad |\Omega| \geq \pi/T. & (2.9) \end{aligned}$$

When these two conditions hold, there is a very simple expression for the impulse response of the discrete time system, $h[n]$, which we are trying to construct, in terms of the impulse response of the continuous time system $h_c(t)$, which we know, and whose effect we would like to mimic. Namely,

$$h[n] = Th_c(nT). \tag{2.10}$$

That is to say, when (2.8)-(2.9) hold, we can simply obtain the discrete-time impulse response $h[n]$, by *sampling* and scaling the desired continuous time impulse response $h_c(t)$. To understand why this is true, let us look at the equation (2.10) in the Fourier domain. Using the expression we derived above for the DTFT of a sampled signal we obtain

$$\begin{aligned} H(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right) & (2.11) \\ &= H_c\left(j\frac{\omega}{T}\right), \qquad |\omega| < \pi, & (2.12) \end{aligned}$$

where in the last line we have used (2.9). Thus, under the condition that the target continuous-time system has band limited frequency response, we can obtain the impulse response of the equivalent discrete-time system simply by sampling (i.e., (2.10)). This property is known as *impulse invariance*.

3 Processing discrete-time signals in continuous time

In the previous section, we showed how in some situations, we could process continuous time signals using discrete-time tools. We can also consider the complementary situation, in which we start with a discrete-time signal $x[n]$, and then

- **Reconstruct** a continuous time signal $x_c(t)$, by ideal bandlimited interpolation, with some assumed sampling period T.
- **Filter in continuous time** by applying some continuous-time system with frequency response $H_c(j\Omega)$ to $x_c(t)$, to form $y_c(t) = x_c * h_c(t)$.
- **Resampling** the signal $y_c(t)$, with sampling period T, to form a final discrete-time signal $y[n]$.

This sequence of operations is not commonly used in practice, but can be helpful for understanding the behavior of certain discrete-time systems.

Notice that because $x_c(t)$ is produced by ideal sinc interpolation, $X_c(j\Omega)$ is bandlimited, with bandlimit π/T:

$$X_c(j\Omega) = \begin{cases} TX(e^{j\Omega T}) & |\Omega| < \pi/T \\ 0 & \text{else.} \end{cases} \tag{3.1}$$

Thus, $Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$ is *also* bandlimited, with bandlimit π/T. We resample with sampling frequency $\Omega_s = 2\pi/T$ at least twice the bandlimit, and so there is no aliasing. Thus, we have

$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{T}Y_c(j\omega/T) \qquad -\pi < \omega \leq \pi, & (3.2)\\ &= \frac{1}{T}H_c(j\omega/T)X_c(j\omega/T) \qquad -\pi < \omega \leq \pi, & (3.3)\\ &= H_c(j\omega/T)X(e^{j\omega}) \qquad -\pi < \omega \leq \pi. & (3.4) \end{aligned}$$

So, *the composite system acts like a discrete-time LTI system with frequency response*

$$H(e^{j\omega}) = H_c(j\omega/T). \tag{3.5}$$

In contrast to the previous section (DT processing of CT signals), here the composite system is *always* LTI. The reason is that because ideal sinc interpolation always produces a bandlimited signal, y_c is automatically bandlimited, and there is no aliasing.

Ex: half-integer delay. As mentioned above, this operation has mostly conceptual / explanatory value. As an example, we can consider the special case of a half-integer delay. Consider a discrete-time system whose frequency response is

$$H(e^{j\omega}) = e^{-j\omega}. \tag{3.6}$$

This corresponds to a delay by a single sample: $y[n] = x[n-1]$.

Now consider a system with frequency response

$$H(e^{j\omega}) = e^{-j\omega/2}. \tag{3.7}$$

It is tempting to consider this a "half-integer delay". But what does this mean? We cannot delay a discrete-time signal by half a sample – "$x[n-1/2]$" doesn't mean anything, because $x[n]$ is only defined for $n \in \mathbb{Z}$.

We can use the inverse DTFT of (3.7) to get some idea: the impulse response of the system is

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-1/2)}d\omega & (3.8)\\ &= \frac{\sin\left(\pi(n-1/2)\right)}{\pi(n-1/2)}. & (3.9) \end{aligned}$$

This impulse response looks like a "shifted sinc" and so we might anticipate that some sort of interpolation is happening here.

To make this interpretation more concrete, notice that this system is equivalent to the following cascade of operations: (i) using ideal sinc interpolation with time period T to form a bandlimited signal $x_c(t)$, (ii) applying a continuous time system whose frequency response satisfies

$$H_c(j\Omega) = H(e^{j\Omega T}) = e^{-j\Omega T/2} \tag{3.10}$$

and then (iii) resampling, with sampling period T. The continuous time system $H_c(j\Omega)$ simply delays a signal by time $T/2$ – half a sampling period. So, our half-integer delay just corresponds to (i) interpolating, (ii) delaying by half a sample in continuous time, and then (iii) resampling the result.

4 Downsampling: Reducing the effective sampling rate

We next apply the viewpoint of sampling to study two operations in which we change the effective sampling rate. These operations are *downsampling*, in which we decrease the effective sampling rate, and *upsampling*, in which we increase the effective sampling rate.

In this section, we describe downsampling. This operation takes a discrete time signal $x[n]$, and produces a new discrete-time signal

$$x_d[n] = x[nM], \tag{4.1}$$

where $M \in \mathbb{Z}_+$ is the *downsampling factor*.[3] This corresponds to a reduction of the effective sampling rate by a factor of M: if the original signal $x[n]$ was generated from some continuous-time signal via $x[n] = x_c(nT)$, the new signal is $x_d[n] = x_c(nMT)$. Thus we have effectively increased the sampling period from T to MT.

Recall that

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j \left[\frac{\omega}{T} - \frac{2\pi k}{T} \right] \right). \tag{4.2}$$

We can build an analogous expression for $X_d(e^{j\omega})$:

$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_{k=-\infty}^{\infty} X_c\left(j \left[\frac{\omega}{MT} - \frac{2\pi k}{MT} \right] \right). \tag{4.3}$$

Notice that any integer k can be written in a unique way as

$$k = \ell + rM, \qquad \text{with} \quad \ell \in \{0, 1, \ldots, M-1\}, \ r \in \mathbb{Z}. \tag{4.4}$$

So, we can rewrite the summation over all k in (4.3) as a double summation – over $r \in \mathbb{Z}$ and $\ell \in \{0, 1, \ldots, M-1\}$. Doing so, we obtain

$$\begin{aligned} X_d(e^{j\omega}) &= \frac{1}{M} \sum_{\ell=0}^{M-1} \frac{1}{T} \sum_{r=-\infty}^{\infty} X_c\left(j \left[\frac{(\omega - 2\pi\ell)/M}{T} + \frac{2\pi r}{T} \right] \right) && (4.5) \\ &= \frac{1}{M} \sum_{\ell=0}^{M-1} X(e^{j(\omega-2\pi\ell)/M}). && (4.6) \end{aligned}$$

Thus, *in frequency domain, downsampling by a factor of M produces M aliased copies of the DTFT $X(e^{j\omega})$. These copies are produced by (i) stretching the frequency axis by a factor of M, and then (ii) shifting by $2\pi\ell$.* Please visualize this through a picture!

[3] The text uses the term "downsampling" in a slightly more general way, to refer to this operation, *together with* a low-pass filtering operation, which eliminates aliasing.

If the original signal $x[n]$ was bandlimited, such that $X(e^{j\omega})$ vanished for $\frac{\pi}{M} \leq |\omega| \leq \pi$, there would be no aliasing. However, in general, there will be aliasing. To avoid aliasing, we need to first lowpass filter, and then downsample. We can avoid aliasing by first passing $x[n]$ through an ideal low-pass filter with cutoff π/M:

$$H_d(e^{j\omega}) = \begin{cases} 1 & |\omega| < \pi/M, \\ 0 & \pi/M \leq |\omega| \leq \pi. \end{cases} \tag{4.7}$$

In practice, due to the unfavorable properties of $h_d[n]$ (infinite impulse response, slow decay), we do not apply this idealized filter. However, it is still important to smooth the signal with some kind of approximate low-pass filter before downsampling. This mitigates the effect of aliasing.

5 Upsampling: Increasing the effective sampling rate

We next look at operations that *increase* the effective sampling rate. Let L be a positive integer. *Upsampling* by a factor of L produces a new signal $x_u[n]$ from $x[n]$, via the expression

$$x_u[n] = \begin{cases} x[n/L] & n = kL, \quad k \in \mathbb{Z} \\ 0 & \text{else.} \end{cases} \tag{5.1}$$

that is to say, we expand the signal $x[\ell]$ by placing the ℓ-th sample at location ℓL in x_u, and filling in zeros in between.

This operation has a simple expression in frequency domain:

$$X_u(e^{j\omega}) = \sum_{n \in L\mathbb{Z}} x[n/L] e^{-j\omega n} \tag{5.2}$$

$$= \sum_{\ell \in \mathbb{Z}} x[\ell] e^{-j\omega L \ell} \tag{5.3}$$

$$= X(e^{j\omega L}). \tag{5.4}$$

So, *upsampling contracts the DTFT by a factor of* L. This can cause aliasing in the high frequencies. To remove aliasing, we can low-pass filter x_u, by applying an ideal low-pass filter

$$H_u(e^{j\omega}) = \begin{cases} L & |\omega| < \pi/L \\ 0 & \pi/L \leq |\omega| \leq \pi. \end{cases} \tag{5.5}$$

6 Optional Appendix: Fourier Transform of the Dirac Comb $s(t)$

Note: this appendix is optional! Sometimes questions arise regarding the continuous-time Fourier relationship

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \xrightarrow{\text{FT}} S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s). \tag{6.1}$$

Here $s(t)$ is the so-called "Dirac comb" – an infintely long sequence of equally spaced Dirac measures. This interesting Fourier relationship indicates that the Fourier transform of a comb is itself a comb – with spikes located at integer multiples of the sampling frequency $\Omega_s = 2\pi/T$. This relationship can be justified on intuitive grounds via the Fourier series. Notice that $s(t)$ is a periodic measure, with period T. Hence, $s(t)$ can be expressed in terms of a Fourier series

$$s(t) = \sum_{k=-\infty}^{\infty} c_k \exp\left(\frac{2\pi k t}{T}\right), \tag{6.2}$$

with

$$c_k = \frac{1}{T}\int_{t=-T/2}^{T/2} s(t) \exp\left(-\frac{2\pi k t}{T}\right) dt \tag{6.3}$$

Noting that on the interval $[-T/2, T/2]$, $s(t) = \delta(t)$, we can see that $c_k = 1/T$ for all k, whence

$$s(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \exp\left(\frac{2\pi k t}{T}\right). \tag{6.4}$$

Using the Fourier relationship

$$\exp\left(\frac{2\pi k t}{T}\right) \xrightarrow{\text{FT}} 2\pi\delta\left(\Omega - \frac{2\pi k}{T}\right) = 2\pi\delta\left(\Omega - k\Omega_s\right). \tag{6.5}$$

we obtain the relationship (6.2).